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The Distribution of the First Return Time for Rational Maps

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We obtain exponential error estimates for the approximation of the zeroth return time to the Poisson distribution for rational maps which might have critical points within the Julia set.

KEY WORDS: Return times; rational maps; Poisson distribution.

1. INTRODUCTION

Recently there has been some great interest in studying the rates of mixing in dynamical systems and how this translates in the distribution and convergence of return times. A rather general result of Galves and Schmitt[(4)] establishes the Poisson distribution of the zeroth return time for a general class of dynamical systems, namely those that are ϕ-mixing. They moreover provide error terms and used this to show in a follow up paper[(1)] that repetition times for subshifts of finite types are normal distributed. For subshifts of finite type Pitskel[(9)] proved that return times of all orders are in the limit Poisson distributed, but he does not give any error terms. Using approximations of transfer operators Hirata[(7, 8)] shows similar results for Axiom A maps. With respect to weaker mixing maps, Poisson distributed return times have been announced by Hirata, Saussol, and Vaienti for a one parametric family of interval maps with an indifferent fixed point.

Here we look at rational maps on the Riemann sphere and their equilibrium states on the Julia set. Because of critical points, the mixing properties are weaker than in the cases mentioned above. However, using distortion theorems, it was shown in ref. 3 that the Central Limit Theorem

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applies. We also know that correlations decay exponentially fast[5] and in ref. 6 we proved that return times in the limit are Poisson distributed (for all orders). In this note we restrict ourselves to the zeroth return time and shall provide error terms for its deviation from the exponential distribution (Theorem 1).

Let us consider rational functions and assume that μ is an equilibrium state for a Hölder continuous potential f which has a "supremum gap" $P(f)-\sup f>0$, where $P(f)$ is the topological pressure of f. Without loss of generality one can assume that $P(f)=0$.

Let $T: \mathbf{C}\to\mathbf{C}$ be a rational map of degree $d\geqslant 2$, and denote by J its Julia set. Let $f: J\to\mathbf{R}$ is a Hölder continuous function which satisfies the condition $P(f)-f>0$ ("supremum gap"), where $P(f)$ is the pressure of f. Then there exists an invariant measure μ on J (μ is conformal with respect to $P(f)-f$). The equilibrium state μ has been extensively studied (see, e.g., refs. 2 and 3).

With appropriate branch cuts on the Riemann sphere one can define univalent inverse branches S_n of T^n on quasidisks Ω_n (which have piecewise smooth boundary) for all $n\geqslant 1$. The actual way in which the branch cuts are executed is irrelevant in our context (since we do not use distortion estimates) and below in Lemma 4 we shall use that branch cuts can be done to suit the purpose at hand. We put $\mathscr{A}^n=\{\varphi(J)\colon \varphi\in S_n\}$ for the n-cylinders (for simplicity's sake we write $\varphi(J)$ for $\varphi(J\cap\Omega_n)$). Note that by ref. 2 the "boundary set" $\partial\mathscr{A}^n=\{\varphi(J\cap\partial\Omega_n)\colon \varphi\in S_n\}$ has zero μ-measure, that is $\mathscr{A}^n$ is a measure theoretic partition of J and the "interiors" of its atoms are pairwise disjoint (the interior of $\varphi(J)$ is understood to be $\varphi(J\cap \operatorname{int}(\Omega_n))$).

Denote by $A_n(x)$ an atom in $\mathscr{A}^n$ for which $x\in A_n(x)$, and put χ_n for the characteristic function of $A_n(x)$. ($A_n(x)$ is almost always unique.)

In ref. 6 (Corollary 20) we showed that the return times are in the limit Poisson distributed for all orders, that is

$$\mu(\{y\in J: \xi_t(y)=r\})\to\frac{t^r}{r!}e^{-t} \tag{1}$$

for μ-almost every x, as n tends to infinity, where

$$\xi_t=\sum_{j=0}^{[t/\mu(\chi_n)]}\chi_n\circ T^j$$

is a "random variable" whose value measures the number of times a given point returns to $A_n(x)$ within the normalized time $t/\mu(A_n(x))$.

In this note we address the question how fast the convergence is in the case of the zeroth ($r=0$) return time. If we put

$$\mathcal{N}_t = \{ y \in J : \tau_n(y) > t/\mu(A_n(x)) \}$$

(zero level set of ξ_t), where $\tau_n(y) = \inf\{k \geqslant 0 : T^k y \in A_n(x)\}$ is the return time for the set $A_n(x)$, then by Eq. (1) $\mu(\mathcal{N}_t) \to e^{-t}$ as $n \to \infty$ almost everywhere. This result is based on an application of a theorem of Sevast'yanov. Here however we will use more elementary argument to get the following error estimate.

Theorem 1. There exists a $\varsigma < 1$ and a constant C_1 so that

$$|\mu(\mathcal{N}_t) - e^{-t}| \leqslant C_1 \varsigma^n$$

for every t and for all $x \in \mathcal{J}_n$, where the set $\mathcal{J}_n \subset J$ has μ-measure at least $1 - n\rho^{n/2}$.

2. MIXING RATES FOR RATIONAL MAPS

We shall need some mixing properties for μ which is the equilibrium state for the potential f. Since f has the "supremum gap," the number $\rho = e^{\sup f - P(f)}$ is less than 1. If we put $g_n = e^{f + fT + fT^2 + \cdots + fT^{n-1} - nP(f)}$ then $|g_n \varphi|_\infty \leqslant \rho^n$. Moreover if $A = \varphi(J)$, $\varphi \in S_n$, is an n-cylinder then T^n is one-to-one on it and, using the fact that μ is $e^{f - P(f)}$-conformal we obtain the following estimate which we shall use several times:

$$\mu(A) = \int_J g_n \varphi \, d\mu \leqslant \rho^n$$

Lemma 2. Let $\kappa > 1$. Then there exists a constant C_2 and $\sigma < 1$ so that

$$|\mu(A \cap T^{-k-n}Q) - \mu(A)\,\mu(Q)| \leqslant C_2 \sigma^k \kappa^n \mu(Q)\,|g_n \varphi|_\infty$$

for all $k, n > 0$, measurable Q and atoms $A = \varphi(J)$ of $\mathscr{A}^n$, where φ is a suitable inverse branch of T^n.

From now on let κ be so that $\kappa\sqrt{\rho} \leqslant 1$ and $\kappa\sqrt{\sigma} \leqslant 1$.

Let us note that if instead of the supremum norm on the right hand side one wants to estimate in terms of the measure of A, then one generally can not control the expanding term κ^n so well and make it grow at an arbitrarily slow exponential rate. If for instance one allows D to be a union of atoms of $\mathscr{A}^n$ (not just contracting ones), then the corresponding mixing property is

$$|\mu(D \cap T^{-k-n}Q) - \mu(D)\,\mu(Q)| \leqslant C_2 \sigma^k \nu^n \mu(D)\,\mu(Q)$$

where $\nu > 1$ is determined by f, although if one only considers contracting branches, then ν can be replaced by κ. In either case one cannot achieve the ϕ-mixing property (which would require the coefficients on the right hand side to decay to zero independently of the "cylinder length" n).

Let $0 < p < 1$ be so that $d^p \sqrt{\rho} \leqslant 1$. In the next lemma we show that those cylinders $A \in \mathscr{A}^n$ that return "to soon" to themselves constitute a small set. Define

$$\mathscr{I}_n^c = \bigcup_{A \in \mathscr{A}^n} \bigcup_{m=1}^{[pn]} A \cap T^{-m}A$$

and then put $\mathscr{I}_n$ for its complement.

Lemma 3.

$$\mu(\mathscr{I}_n^c) \leqslant n\rho^{n/2}$$

Proof. Let τ_φ denote the first return time to the set A_φ, $\varphi \in S_n$ and define

$$U_m = \{ y \in J : \tau_\varphi(y) = m\}$$

and obtain

$$U_m \cap A_\varphi \subseteq A_\varphi \cap T^{-m}A_\varphi \subseteq \bigcup_{k=0}^{m} U_k \cap A_\varphi$$

With $V = T^m U_m \cap A_\varphi$ we have $V = A_\varphi \cap T^m A_\varphi$. Let us write $\varphi = \psi^1 \varphi^1$, where $\psi^1 \in S_m$ and $\varphi^1 = T^m \varphi \in S_{n-m}$ (with suitable branch cuts). We proceed inductively and obtain

$$\varphi = \psi^k \psi^{k-1} \cdots \psi^1 \varphi^k$$

where $n = mk + l$, $0 \leqslant l \leqslant m$, $\psi^j \in S_m$ and $\varphi^k = T^{mk}\varphi \in S_l$. Let us note that $T^{mj} V = A_{\varphi^j} \cap A_{\varphi^{j+1}}$ for $j = 1,..., k$, where $\varphi^j = T^{jm}\varphi = \psi^{j+1} \cdots \psi^1 \varphi^k$. Since $\mu(A_{\psi^k \cdots \psi^1 \varphi^k}) \leqslant \rho^{n+m}$ we can now estimate

$$\sum_{\varphi \in S_n} \mu(U_m \cap A_\varphi) \leqslant \sum_{\psi^1,..., \psi^k \in S_m} \mu(A_{\psi^k \cdots \psi^1 \varphi^k}) \leqslant |S_m|\; \rho^{n+m}$$

where there are at most $|S_m|$ choices for ψ^1 and then for every $j = 1,...,$ $k-1$ the $\psi^{j+1} \in S_m$ must satisfy $T^{jm}V \subset A_{\psi^{j+1}} \cap A_{\psi^j}$. For every ψ^j we get a unique ψ^{j+1} since the sets $\psi(J \cap \text{int}(\Omega_m))$, $\psi \in S_m$ are disjoint. Hence the last inequality, where we also used the fact that $\mu(A_{\tilde{\varphi}}) \leqslant |\varphi|_\infty \leqslant \rho^{n+m}$ for $\tilde{\varphi} \in S_{n+m}$.

Since by assumption $d^p \sqrt{\rho} \leqslant 1$ we get

$$\sum_{\varphi \in S_n} \mu(U_m \cap A_\varphi) \leqslant d^m \rho^{n+m} \leqslant (d^p \rho^{1/2})^n \rho^{n/2} \rho^m \leqslant \rho^{n/2}$$

and therefore

$$\mu(\mathscr{I}_n^c) \leqslant \sum_{m=0}^{[pn]} \sum_{\varphi \in S_n} \mu(U_m \cap A_\varphi) \leqslant n \rho^{n/2}$$

which goes to zero as n goes to infinity. ∎

3. PROOF OF THE MAIN THEOREM

Put $h(r) = \mu(\mathscr{N}_r)$, where $\mathscr{N}_r$ is the zero level set of ξ_r. For simplicity put $A = A_n(x)$ and $\mathscr{M}_r = J \setminus \mathscr{N}_r = \{ y \in J : \xi_r(y) > 0 \}$. We immediately obtain the upper bound $\mu(\mathscr{M}_r) \leqslant r + \mu(A)$ and a lower bound in the following lemma. For the next lemma we will require that $x \in \mathscr{I}_n$.

Lemma 4. Assume $x \in \mathscr{I}_n$. Then there exists an $\eta < 1$ and a constant C_3 so that

$$\mu(\mathscr{M}_r) \geqslant r(1 - C_3 \eta^n)$$

Proof. Let $A = A_n(x)$, put $B_0 = A$ and define for $j = 1,...,$ $[r/\mu(A)]$

$$\begin{aligned} B_j &= T^{-j}A \Big\backslash \bigcup_{l=0}^{j-1} (T^{-j}A \cap T^{-l}A) \\ &\subseteq T^{-j} \left(A \Big\backslash \bigcup_{l=0}^{j-1} (A \cap T^{-l+j}A) \right) \end{aligned}$$

Since $\mathscr{M}_r$ is the disjoint union of B_j, we get by invariance of the measure

$$\mu(B_j) \geqslant \mu(A) - \sum_{l=1}^{j} \mu(A \cap T^{-l}A)$$

Since by assumption $x \in \mathscr{J}_n$ we have $A \cap T^{-l}A = \emptyset$ for $l \leqslant pn$, and obtain

$$\mu(B_j) \geqslant \mu(A) - \sum_{l=[pn]+1}^{j} \mu(A \cap T^{-l}A)$$

To estimate $\mu(A \cap T^{-l}A)$ for $l \in ([pn], n)$ note that T^l is one-to-one on A. Thus, if we arrange for suitable branchcuts, we can find an inverse branch ψ of T^l so that $A \subseteq A_\psi = \psi(J)$ and estimate according to Lemma 2 as follows

$$\mu(A \cap T^{-l}A) \leqslant \mu(A_\psi \cap T^{-l}A) \leqslant (1 + C_2)\, \mu(A)\, |g_l \psi|_\infty \leqslant c_1\, \mu(A)\, \rho^l$$

For $l > n$, we get again by Lemma 2

$$\mu(A \cap T^{-l}A) \leqslant \mu(A)(\mu(A) + C_2 \sigma^{l-n} \kappa^n\, |g_n \varphi|_\infty)$$

where $\kappa > 1$ can be chosen arbitrarily and $C_2 = C_2(\kappa)$ is independent of n and l. Since $|g_n \varphi|_\infty \leqslant \rho^n$ we can pick $\kappa = 1/\sqrt{\rho}$ to achieve

$$\mu(A \cap T^{-l}A) \leqslant \mu(A)(\mu(A) + C_2 \sigma^{l-n} \rho^{n/2}) \leqslant c_2\, \mu(A)\, \sigma^{l-n} \rho^{n/2}$$

Thus, for $j \geqslant 1$ (assume $p \leqslant 1/2$ and $\sigma \leqslant \rho^p$):

$$\begin{aligned}\mu(B_j) &\geqslant \mu(A) - \sum_{l=n+1}^{\infty} c_2\, \mu(A)\, \sigma^{l-n} \rho^{n/2} - \sum_{l=[pn]+1}^{n} c_1\, \mu(A)\, \rho^l \\ &\geqslant \mu(A)(1 - c_3\, \rho^{pn})\end{aligned}$$

and since $\mu(B_0) = \mu(A)$ we get

$$\begin{aligned}\mu(\mathscr{M}_r) &= \sum_{j=0}^{[r/\mu(A)]} \mu(B_j) \\ &\geqslant \left(\left[\frac{r}{\mu(A)} \right] + 1 \right) \mu(A)(1 - c_3\, \rho^{pn}) \\ &\geqslant r(1 - c_3 \eta^n)\end{aligned}$$

where $\eta = \rho^p$ and $C_3 = c_3$. The statement of the lemma follows. ∎

We obtain the following multiplicative type property for the function h.

Lemma 5. There exists a constant C_4 so that for all $t, r > 0$ and all n large enough

$$|h(t+r) - h(t)\, h(r)| \leqslant C_4\, \rho^{n/2}$$

Proof. Let us first note that

$$\mathcal{N}_{t+r} = \mathcal{N}_{r-k} \cap T^{-(R-K)} \mathcal{N}_k \cap T^{-R} \mathcal{N}_t$$

where $R = [r/\mu(A)]$, $k = K\mu(A)$ and $K \geqslant n$ is some numbers so that $R-K$ is positive, which is possible if n is large enough. Thus, by T-invariance of μ,

$$|\mu(\mathcal{N}_{t+r}) - \mu(\mathcal{N}_{r-k} \cap T^{-R} \mathcal{N}_t)| \leqslant \mu(\mathcal{M}_k) \tag{2}$$

where a rough estimate yields

$$\mu(\mathcal{M}_k) \leqslant K\mu(A)$$

and similarly

$$|\mu(\mathcal{N}_r) - \mu(\mathcal{N}_{r-k})| \leqslant \mu(\mathcal{M}_k) \leqslant K\mu(A) \tag{3}$$

Next we use the mixing property of μ. Note that

$$\mathcal{N}_{r-k} = J \Big\backslash \bigcup_{j=0}^{R-K} T^{-j} A$$

and therefore

$$\begin{aligned}
\mu(\mathcal{N}_{r-k} \cap T^{-R} \mathcal{N}_t) &= \mu\left(\left(J \Big\backslash \bigcup_{j=0}^{R-K} T^{-j} A\right) \cap T^{-R} \mathcal{N}_t\right) \\
&= \mu(\mathcal{N}_t) - \mu\left(\bigcup_{j=0}^{R-K} T^{-j} A \cap T^{-R} \mathcal{N}_t\right)
\end{aligned}$$

while on the other hand one has

$$\mu(\mathcal{N}_t)\, \mu(\mathcal{N}_{r-k}) = \mu(\mathcal{N}_t)\left(1 - \mu\left(\bigcup_{j=0}^{R-K} T^{-j} A\right)\right).$$

Hence (the inverse branch φ of T^n is so that $A=\varphi(J)$) an application of Lemma 2 yields

$$\begin{aligned}
&|\mu(\mathcal{N}_{r-k}\cap T^{-R}\mathcal{N}_t)-\mu(\mathcal{N}_t)\,\mu(\mathcal{N}_{r-k})|\\
&\quad=\left|\mu\left(\bigcup_{j=0}^{R-K}T^{-j}A\cap T^{-R}\mathcal{N}_t\right)-\mu(\mathcal{N}_t)\,\mu\left(\bigcup_{j=0}^{R-K}T^{-j}A\right)\right|\\
&\quad\leqslant\sum_{j=0}^{R-K}|\mu(T^{-j}A\cap T^{-R}\mathcal{N}_t)-\mu(\mathcal{N}_t)\,\mu(T^{-j}A)|\\
&\quad=\sum_{j=0}^{R-K}|\mu(A\cap T^{-(K+j)}\mathcal{N}_t)-\mu(\mathcal{N}_t)\,\mu(A)|\\
&\quad\leqslant C_2\sum_{j=0}^{\infty}\kappa^n\sigma^j\mu(\mathcal{N}_t)\,|g_n\varphi|_\infty\\
&\quad\leqslant c_1\,\rho^{n/2}
\end{aligned}$$

since $K\geqslant n$, where we used that $\mu(\mathcal{N}_t)\leqslant 1$, $\mu(A)\leqslant\rho^n$ and $\kappa\sqrt{\rho}\leqslant 1$. This estimate combined with Eqs. (2) and (3) yields by the triangle inequality

$$\begin{aligned}
&|h(t+r)-h(t)\,h(r)|\\
&\quad\leqslant|\mu(\mathcal{N}_{t+r})-\mu(\mathcal{N}_{r-k}\cap T^{-R}\mathcal{N}_t)|\\
&\qquad+|\mu(\mathcal{N}_{r-k}\cap T^{-R}\mathcal{N}_t)-\mu(\mathcal{N}_t)\,\mu(\mathcal{N}_{r-k})|+|\mu(\mathcal{N}_r)-\mu(\mathcal{N}_{r-k})|\\
&\quad\leqslant c_1\,\rho^{n/2}+2K\mu(A)\\
&\quad\leqslant C_4\,\rho^{n/2}\qquad\blacksquare
\end{aligned}$$

For suitable choice of K the statement of Lemma 5 can be improved to $|h(t+r)-h(t)\,h(r)|\leqslant\text{const}\,\mu(A)^\alpha$ for any $\alpha<1$.

By an induction argument one now obtains (cf. ref. 4, Lemma 6):

$$|h(kr)-h(r)^k|\leqslant\frac{C_4\,\rho^{n/2}}{1-h(r)}\tag{4}$$

Proof of Theorem 1. Put $A=A_n(x)$ and let us now estimate $h(r)^k-e^{-t}$, where we put $t=kr$, $k\geqslant 1$. By Lemma 4

$$h(r)=1-\mu(\mathcal{M}_r)\leqslant 1-r+rC_3\eta^n$$

and thus

$$\begin{aligned} h(r)^k - e^{-t} &\leqslant (1 - r + rC_3\eta^n)^k - e^{-t} \\ &\leqslant e^{k(-r + rC_3\eta^n)} - e^{-t} \\ &= e^{-t}(e^{krC_3\eta^n} - 1) \\ &\leqslant 2e^{-t}tC_3\eta^n \end{aligned}$$

if $krC_3\eta^n$ is small enough (say $\leqslant 1/2$). The lower bound is done similarly:

$$\begin{aligned} e^{-t} - h(r)^k &\leqslant e^{-t} - (1 - r - \mu(A))^k \\ &\leqslant e^{-t} - e^{-k(r+\mu(A)) - k(r+\mu(A))^2} \\ &\leqslant e^{-t}(k\mu(A) + k(r+\mu(A))^2) \end{aligned}$$

for $r + \mu(A)$ small enough. Thus

$$|h(r)^k - e^{-t}| \leqslant c_1 t\eta^n e^{-t}$$

Now let us pick $r \in (\rho^{n/6}, 2\rho^{n/6})$ so that $k = t/r$ is an integer. We obtain using Eq. (4) and Lemma 4 (recall that $|1 - h(r)| \geqslant \text{const } r$):

$$\begin{aligned} |h(t) - e^{-t}| &\leqslant |h(t) - h(r)^k| + |h(r)^k - e^{-t}| \\ &\leqslant c_2 \frac{\rho^{n/2}}{1 - h(r)} + c_1 t\eta^n e^{-t} \\ &\leqslant c_3 \rho^{n/3} + c_1 t\eta^n e^{-t} \\ &\leqslant C_1 \varsigma^n \end{aligned}$$

for $\varsigma < \min(\rho^{1/3}, \eta)$. ∎

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